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Constraints, positivity, 'obscure' representations and the spin-$\frac{3}{2}$ problem

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Abstract. We trace the failure of Einstein causality in interacting high-spin field theories to the presence of secondary constraints in the Cauchy initial-value problem. We bring in and extend a theorem of Gel'Fand and Yaglom, and use it to show that non-negativity of the field anticommutators for half-integral spin, and non-negativity of the field energy for integral spin, necessarily imply the existence of such constraints, except for the privileged spin values 0, $\frac{1}{2}$ and 1. We prove this under very lax assumptions as to the nature of the mass spectrum.

We carry out a systematic search for alternative spin-$\frac{3}{2}$ constraint schemes, taking a Lagrangian which contains thirteen adjustable parameters, many of which couple in the hitherto 'obscure' unmixed-spinor representations $(\frac{3}{2}, 0)$ and $(0, \frac{3}{2})$ of the Lorentz group. The search reveals two unfamiliar spin-$\frac{3}{2}$ equations, one of which is entirely new. We analyse the new equations and find that each contains a Rarita–Schwinger constraint subchain. This means that the new equations will also show causality failures.

We find that in five or more space–time dimensions additional field equations appear, which are very much simpler and give a much more fundamental role to the 'obscure' representations. We infer that our inability to find a viable alternative to the Rarita–Schwinger scheme reflects an essential mathematical feature of four-dimensional space–time, and that it is not due just to a lack of imagination. We examine the dimension dependence of the compound-spin equation of Harish-Chandra, and find further support for this assessment. We conclude that the only acceptable dynamical equations in four-dimensional Minkowski space–time are those customarily associated with spins 0, $\frac{1}{2}$ and 1.

1. Introduction

It is well known that no satisfactory Lagrangian field equation for charged particles of spin $\frac{3}{2}$ has ever been discovered. Although the Rarita–Schwinger equation, namely

$$(\gamma\partial)\psi^\mu - \partial^\mu(\gamma\psi) - \gamma^\mu(\partial\psi) - \gamma^\mu(\gamma\partial)(\gamma\psi) + m\psi^\mu + m\gamma^\mu(\gamma\psi) = C^\mu, \qquad (1)$$

$(g^{00} = 1 = -\gamma^0\gamma^0)$ works perfectly well in the presence of *any* kinematically independent vector–spinor source C^μ, it gives superluminal propagation for sources C^μ depending on ψ itself and on various interaction potentials. This unpleasant and puzzling phenomenon has generated a vast literature (see Allcock and Hall 1977, Cox 1976, 1977 for references).

The cause of the trouble may be reliably diagnosed by referring to the non-covariant Cauchy initial-value problem. The component of (1) with $\mu = 0$ is free of

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time derivatives, and thus imposes a *primary constraint* on the Cauchy initial-value data. In the simple case of a zero-momentum free field this constraint asserts that $\gamma_i\psi^i$ vanishes; more generally various spatial derivatives of ψ^i also contribute. A *secondary constraint* on the data arises by considering the time derivative of the primary constraint, and in the simple case just mentioned it asserts that ψ^0 vanishes. But when interaction is present the spatial derivatives of $\dot{\psi}^i$ in the once-differentiated primary constraint necessarily induce spatial derivatives of the interaction potentials in the resulting secondary constraint, and as might be expected these modify the value of the determinant which one encounters when solving the secondary constraint for the redundant field components. In the particular case of the electromagnetic coupling it is found that the determinant acquires in this way a dependence upon the magnetic field $\mathbf{B}$, and that it vanishes if $B = 3m^2/2e$. Unless the electromagnetic field tensor is altogether zero there will always be some inertial frames of reference within which B assumes this critical value, and in these frames the Cauchy problem cannot be set. That is, propagation is instantaneous in these frames. This implies superluminal propagation in any frame, since (1) is Lorentz invariant.

It is thus clear that the trouble can be traced directly to the presence of secondary constraints, the analysis of which brings in derivatives of the interaction potentials. Such derivatives, if non-zero, are always likely to be dangerous, since they can be made as big as we please by changing the reference frame. Obviously a safe way to avoid acausality troubles would be to use equations free of secondary constraints (Amar and Dozzio 1975, Cox 1976, Prabhakaran *et al* 1977). There is however a group-representation theorem of Gel'Fand and Yaglom which, in combination with other established results, completely blocks this escape route. We explain the technical details of this application of the Gel'Fand–Yaglom theorem in § 2; the upshot is that non-negativity of the field anticommutators in Lagrangian systems uncontaminated by massless modes or gauge freedoms necessarily requires the presence of secondary constraints whenever there are modes with half-integral spin higher than $\frac{1}{2}$. This is a strong result, since it does not depend upon any of the customary assumptions as to Klein–Gordon conditions, etc. It is probable that it could be still further strengthened, since in practice it is found necessary to bring in quaternary constraints for spin $\frac{5}{2}$, sexenary constraints for spin $\frac{7}{2}$, and so forth (Amar and Dozzio 1972*a*, *b*, Singh and Hagen 1974, Giesen 1975, Cox 1978).

The Gel'Fand–Yaglom theorem does not leave much room for manoeuvre. The effect of non-minimal electromagnetic coupling terms in the Lagrangian has already been tried by Shamaly and Capri (1972) and, independently, by one of us (Hall 1976), but to no avail. The non-minimal terms move the critical point, but cannot get rid of it. Hall (1976) has also investigated the general conditions required of any field-dependent modifications of the coefficients in the *kinetic* part of the Lagrangian, and has found that all simple electromagnetic insertions violate them. Massive gauge theories can also be invented—one such comes by varying the massive Rarita–Schwinger Lagrangian under the covariant condition $(\partial\psi)+(\gamma\partial)(\gamma\psi)=0$—but they only make matters worse. As in the massless spin-$\frac{3}{2}$ theory (Allcock and Hall 1977) the requisite spinor gauge principles cannot be maintained in the presence of electromagnetism.

There remains only the hope that one might perhaps find some entirely new equation in which the secondary constraints would arise in a different way, more amenable to the electromagnetic interaction. One would expect that if there were such a new equation, it would exploit the 'obscure' representations $(\frac{3}{2}, 0)$ and $(0, \frac{3}{2})$ of the Lorentz group, rather than the representations $(1, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ of the Rarita–Schwinger or the equivalent Fierz–Pauli scheme. Our main purpose in writing this paper is to report upon work we have done in this direction and, as it turns out, to provide thereby a rather strong and convincing argument that this last hope is illusory. One of the difficulties which faced us in this part of the work was that a search for new equations is not limited by any known mathematical principle, but only by the amount of complication deemed to be tolerable. We consider nevertheless that our search programme (described in § 3 and supplemented by further discussion in §§ 4 and 5) provides sufficient circumstantial evidence in the case. In § 6 we conclude therefore that particles of spin higher than 1 cannot, even in principle, be considered as elementary.

2. The strengthened Gel'Fand–Yaglom theorem

In this section we discuss the quantisation of the general Gel'Fand–Yaglom equation for particles of half-integral spin. Using real field variables ψ for convenience, as is always possible, and restricting our attention to spatially uniform initial-value data, we deal with the equation

$$A\dot{\psi} + B\psi = 0. \tag{2}$$

The real matrices A and B are symmetric and antisymmetric respectively, the vector ψ is anticlassical (i.e. its elements mutually anticommute—for a more precise specification see Allcock 1975a), and the appropriate anticlassical Lagrangian is

$$L = \tfrac{1}{2}\mathrm{i}\psi(A\dot{\psi} + B\psi). \tag{3}$$

The matrix B is non-singular by hypothesis, because otherwise there would be massless modes and/or gauge freedoms. The matrix A is however allowed to be singular, and to each of its linearly independent null vectors v there is a corresponding primary constraint $vB\psi = 0$.

The group-theoretic property which we need is formulated by Gel'Fand and Yaglom in classical and complex terms, but we can easily establish contact by bringing in a coomplexified version of (2), in which both (2) itself and its Lagrangian (3) are modified by replacing the anticlassical real field ψ by a classical complex field Ψ, with conjugation of the first factor Ψ then appearing in (3). We point out that the Lagrange bracket which appears in the anticlassical variational calculus (Allcock 1975a) and in the invariant quantisation theory (Allcock 1975b, Allcock and Hall 1977), namely the anticlassical bilinear form

$$\tfrac{1}{2}\mathrm{i}(\psi_1 A\psi_2 - \psi_2 A\psi_1) \equiv \tfrac{1}{2}(\psi_1 + \mathrm{i}\psi_2)^* A(\psi_1 + \mathrm{i}\psi_2) \tag{4}$$

evaluated on two anticommutative solutions ψ_1 and ψ_2 of (2), is governed by the same matrix A as appears in the classical 'charge'

$$Q \equiv \Psi_1 A\Psi_1 + \Psi_2 A\Psi_2 \equiv \Psi^* A\Psi \tag{5}$$

evaluated on a complex classical solution $\Psi \equiv \Psi_1 + \mathrm{i}\Psi_2$.

To apply the Gel'Fand–Yaglom theorem we need two lemmas, which we now state together with the theorem itself. The condition $\det B \neq 0$ is understood to be operative throughout.

Lemma 1. Non-negativity of the anticommutators of any quantised Gel'Fand–Yaglom field ψ implies positive definiteness of the same, and implies also positive definiteness of the classical 'charge' Q in the space of the complex classical solutions Ψ, and conversely.

Lemma 2. If a Gel'Fand–Yaglom equation has no secondary constraints, then positive definiteness of the classical 'charge' Q in the space of the complex classical solutions Ψ implies and is implied by non-negativity of Q for any vector Ψ whatever.

Theorem. (Gel'Fand and Yaglom 1948, Gel'Fand *et al* 1963, Naimark 1964). If the Hermitian form Q is finite dimensional, and if it is non-negative for any Ψ whatever, then it is impossible for any associated Gel'Fand–Yaglom equation to carry spins higher than $\frac{1}{2}$.

Following through lemmas 1 and 2 and the theorem we see that any finite-dimensional Gel'Fand–Yaglom system having non-negative anticommutators and carrying spins higher than $\frac{1}{2}$ must perforce generate secondary constraints. Infinite-dimensional systems can avoid this problem, but their mass spectra (Naimark 1964) render them wholly unsuitable for physical application.

Proof of Lemma 1. Consider the constraints which arise in the linear analysis of the equation (2) or its complexified classical counterpart. By exploiting them to the full to eliminate as many redundant components of $\psi(\Psi)$ as possible we may write the anticlassical Lagrange bracket (4) and the classical 'charge' (5) in the respective reduced forms

$$\tfrac{1}{2}(\psi_1' + \mathrm{i}\psi_2')^* A'(\psi_1' + \mathrm{i}\psi_2'), \tag{6}$$

$$\Psi'^* A' \Psi', \tag{7}$$

where A' is a symmetric real matrix of suitably reduced dimension. We make the same reduction in the Lagrangian itself and vary the reduced action with respect to ψ' (Ψ'). We then obtain the reduced equation of motion

$$A'\dot{\psi}' + B'\psi' = 0, \tag{8}$$

with a corresponding equation for Ψ'. It is obvious that these reduced equations must be compatible with the full equations (2), since they are based on a less extensive set of variations.

We now invoke the theorem of rank of the calculus of variations (Allcock 1975a, theorem 6), which asserts that the fully constrained variation of the action of the deterministic system (3) or its classical counterpart produces the same equations of motion (albeit mixed together) as does the unconstrained variation. This theorem tells us that the reduced equation of motion (8) is entirely equivalent to (2), modulo the constraints, and not merely compatible. Therefore there cannot be any massless modes or gauge freedoms in (8), and from that we must infer that our procedure for the construction of a reduced system automatically presents us with a non-singular reduced matrix B'. This in turn implies that A' is also automatically non-singular, since all constraints have already been accounted. We have thus obtained an equivalent constraint-free and non-degenerate reduced Lagrangian for the rest frame. The reduced Lagrange bracket (6) and the reduced classical 'charge' (7) are equal to those displayed in (4) and (5) for all rest-frame solutions, by the very nature of the construction.

We can now use the reduced Lagrangian to study the rest-frame anticommutators. Various simple theories of quantisation (see, e.g., Allcock 1975b, § 1.4) suggest that the anticommutators between the reduced components will be equal to the elements of the symmetric real matrix A'^{-1}, whose existence is assured by the arguments given. The invariant theory of quantisation (Allcock 1975b) tells us that the anticommutator obtained in this way is the correct one. Indeed the invariant theory bases the recipe for quantisation entirely on the Lagrange bracket in solution space and the time evolution, and these are the same for the reduced and unreduced systems.

The anticommutators will thus be non-negative if and only if A'^{-1} is non-negative. But a non-negative symmetric real matrix is of necessity positive definite if it is non-singular. So A'^{-1} must be positive definite, and A' likewise. The connection between this and the positive definiteness of the charge now follows in an obvious way from (7), where the same matrix A' appears. Lemma 1 is thus proved.

A connection between charge signature and anticommutator signature has been noted by many authors (e.g. Harish-Chandra 1946). The present treatment differs from others that have been given, in that it establishes invertibility of the relevant matrix A' without invoking any restrictive assumptions, such as the assumption that every solution of the equation obeys a Klein–Gordon condition. Thus it covers multi-mass equations, including those with mass degeneracies.

Proof of lemma 2. We use all the concepts introduced in the proof of lemma 1 and, in accordance with that lemma and the positivity principle, we now limit our attention to cases where A' is positive definite and all constraints are primary. Let the full matrix A have rank R, nullity N, and dimension $R+N$. Then N linearly independent primary constraints are generated, and for the purposes of lemma 2 these are the only constraints. The non-singular reduced matrix A' obtained by restricting ψ (Ψ) to the constrained subspace thus has its dimension, and hence also its rank, equal to R (it would be less than this if secondary constraints were present). Imagine now that the Hermitian form Q had been brought to diagonal by the usual Lagrange algorithm of completion of squares, before making the reduction. Q would then be expressed explicitly in terms of the squares of precisely R linearly independent linear combinations of the components of Ψ. Call these combinations the ranking variables. If any of the N constraints or some linear combination thereof were to involve only the ranking variables then the reduction to the constrained subspace would reduce the rank below R. But we have just proved that this does not happen. Therefore we must conclude that the constraints serve merely to express the N non-ranking variables in terms of the R ranking variables. So if Q is already expressed in terms of ranking variables then reduction manifestly makes no difference to it. Consequently the signature of A is equal to the signature of A'. So if A' is positive definite, then A is non-negative. This proves lemma 2.

Lemma 2 can also be proved by using irrational methods, and although these are less direct we must mention them briefly, in order to clarify the connection with certain diagonalisability conditions imposed by Gel'Fand and Yaglom. Firstly we note that for absence of secondary constraints in a Gel'Fand–Yaglom equation it is both necessary and sufficient that the zero roots of the propagation matrix $B^{-1}A$ be simple or semisimple. Secondly we note that positive definiteness of the charge Q in the solution space of a Gel'Fand–Yaglom equation forces the non-zero roots of $B^{-1}A$ to be simple or semisimple (Udgaonkar 1952, Speer 1969, Wightman 1973, 1978). Thus the combined conditions imposed in the statement of lemma 2 make it necessary that $B^{-1}A$ be diagonalisable. Thirdly, we note that Gel'Fand and Yaglom proved that if $B^{-1}A$ is diagonalisable and if Q is positive definite in the solution space, then Q is non-negative for any Ψ whatever. So we have reached lemma 2 again.

The diagonalisability theorems of Udgaonkar and Speer to which we have just referred hold for any Gel'Fand–Yaglom equation satisfying the non-negativity condition of lemma 1, irrespective of its constraint structure. They do not depend upon prior assumptions as to the nature of the mass spectrum, etc, but only upon the interplay between symmetric and antisymmetric matrices. Indeed, let A' be any symmetric real positive definite matrix and B' any antisymmetric real non-singular matrix. Then $B'^{-1}A'$ is diagonalisable, and furthermore its eigenvalues come in non-zero imaginary conjugated pairs (essentially, this is just the well known theorem that every antisymmetric real matrix can be diagonalised by a unitary transformation—see also Allcock 1978, in preparation). Applying these eigen-properties to the reduced matrices in equation (8) we obtain the following lemma.

Lemma 3. Any Gel'Fand–Yaglom field with non-singular mass matrix B and non-negative anticommutators gives rise to a real mass spectrum and admits a decomposition into simple Fermi–Dirac oscillators.

For boson systems A is antisymmetric and B is symmetric non-singular, and non-negativity of the energy is the essential criterion. Thus B' in (8) is required to be non-negative. Lemma 1 then asserts that both B' and $A'^{\mathrm{T}}B'^{-1}A'$ are positive definite. Lemma 2 is formulated for bosons in the following way. We know that the quadratic form $\dot{\psi}A^{\mathrm{T}}B^{-1}A\dot{\psi}$ is equal to the positive-definite form $\dot{\psi}'A'^{\mathrm{T}}B'^{-1}A'\dot{\psi}'$ within the R-dimensional linear space of all possible constrained $\dot{\psi}$. We know also that the same form $\dot{\psi}A^{\mathrm{T}}B^{-1}A\dot{\psi}$ is equal to zero within the N-dimensional linear space of all null vectors $\dot{\psi}$ of A. The full dimension is $R+N$ (since there are no secondary constraints) and therefore, since rank plus nullity equals dimension, its full rank must also be R, and its signature likewise. Thus $\dot{\psi}A^{\mathrm{T}}B^{-1}A\dot{\psi}$ is non-negative. At this point the Gel'Fand–Yaglom group-theoretic analysis comes into play, limiting the spin values to 0 and 1. If secondary, etc, constraints are allowed we still have lemma 3, which demonstrates the existence of a real mass spectrum and the possibility of a decomposition into simple Bose–Einstein oscillators.

3. A search for new constraint structure

In § 1 we have pointed out that secondary constraints are likely to be dangerous, and in § 2 we have proved that they cannot be avoided. But they do not necessarily spell disaster. Thus Shamaly and Capri (1973) formulated a perfectly consistent and causal spin-1 equation having secondary constraints and non-minimal electromagnetic coupling. Perhaps their ability to do this was linked in some way to the fact that spin 1 can also be described by the ten-component Duffin–Kemmer equation, which has all its constraints primary. Be that as it may, we thought it appropriate to make a search for spin-$\frac{3}{2}$ equations with alternative constraint structure.

There are unfortunately no known principles to guide such a search. On the other hand, it is well known that the covariant *wave mechanics* of a single free particle of spin s can be adequately formulated by using as wavefunction a $(2s+1)$-component positive energy solution of the Klein–Gordon equation, transforming either by the representation with Weyl spinor character $(s, 0)$ or by its complex conjugate $(0, s)$ (Joos 1962, Weinberg 1964, 1969, Niederer and O'Raifeartaigh 1974, Novozhilov 1975). Such a description uses no constraints at all, but unfortunately it cannot be carried over very directly to the quantum field level of description. The reason is that it identifies the geometric or chiral $\surd-1$ of the representation theory of the Lorentz group with the $\surd-1$ of the quantum formalism, even before second quantisation, whereas all techniques of second quantisation bring in the quantal $\surd-1$ as an entirely separate and distinct entity. This leads automatically to a hypercomplex structure in the state space of the quantum field theory, since the two square roots can be independently conjugated. That is why all simple attempts to use $(\frac{3}{2}, 0)$ in a field formalism encountered a doubling of states, and an associated indefinite quantisation metric (Hurley 1972, 1974, Fisk and Tait 1973, Biritz 1975, Seetharaman *et al* 1975, Eeg 1976). It seemed nevertheless quite obvious to us that our search would have to bring in the representation $(\frac{3}{2}, 0)$ in some way. This hunch was further supported by our earlier investigations of the role of that representation in the massless case (Allcock and Hall 1977).

Because of the above doubling problem, and for other good reasons too, we chose to work with a tensor–spinor notation and real Dirac matrices. We implemented our search by taking an anticlassical (real or complex) reducible antisymmetric tensor–spinor $\chi^{\mu\nu} = -\chi^{\nu\mu}$ as the basic field variable (Dirac index suppressed). $\chi^{\mu\nu}$ carries all the representations that might most plausibly be useful. Its γ-irreducible part, namely

$$(\chi^{\mu\nu})_{\mathrm{irr}} \equiv \chi^{\mu\nu} - \tfrac{1}{2}\gamma^{\mu}\gamma_{\rho}\chi^{\nu\rho} + \tfrac{1}{2}\gamma^{\nu}\gamma_{\rho}\chi^{\mu\rho} - \tfrac{1}{12}[\gamma^{\mu}, \gamma^{\nu}]\gamma_{\rho}\gamma_{\sigma}\chi^{\rho\sigma}, \tag{9}$$

transforms by the eight-dimensional real-irreducible complex-reducible real representation $(\frac{3}{2}, 0)\oplus(0, \frac{3}{2})$. The reduced vector-spinor

$$\psi^{\mu} \equiv \gamma_{\rho}\chi^{\mu\rho} \tag{10}$$

carries the twelve- and four-dimensional real-irreducible complex-reducible real representations $(1, \frac{1}{2})\oplus(\frac{1}{2}, 1)$ and $(\frac{1}{2}, 0)\oplus(0, \frac{1}{2})$, as used in the neutral Rarita–Schwinger equation. We proceeded to write down the Lagrangian for the most general possible parity-conserving second-order field equation for a real $\chi^{\mu\nu}$. (Since $\chi^{\mu\nu}$ is real-reducible the definition of the parity operation is partly a matter of convention. But the parity of each real-irreducible constituent can be separately reversed by prefixing a factor γ_5, so we may always adopt the uniform convention $\chi^{0j} \to -\gamma^0\chi^{0j}$, $\chi^{jk} \to \gamma^0\chi^{jk}$). The Lagrangian thus introduced contains three independent real mass coefficients, four independent real coefficients for terms linear in the derivatives of χ (discounting perfect differentials), and six independent real coefficients for terms quadratic in the derivatives of χ (discounting perfect differentials). It is significant (cf § 4) that the numbers of coefficients would be respectively 3, 5 and 7, were it not for the Dirac matrix identity

$$\begin{aligned}(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta})\gamma^{\rho} + \tfrac{1}{2}[\gamma^{\mu}, \gamma^{\nu}](\gamma_{\alpha}\delta^{\rho}_{\beta} - \gamma_{\beta}\delta^{\rho}_{\alpha} - \tfrac{1}{6}\gamma^{\rho}[\gamma_{\alpha}, \gamma_{\beta}])\\ + (\text{cyclic permutations of } \mu, \nu, \rho) \equiv 0 \qquad (D \leqslant 4),\end{aligned} \tag{11}$$

which holds in spaces of dimension $D \leqslant 4$ and thereby limits the possibilities. (Identity

(11) is an algebraic consequence of the fundamental anticommutation rule of the γ-algebra. When $D=4$ it can also be written in terms of γ_5 and $\epsilon^{\mu\nu\rho\sigma}$, a form which is more familiar (MacFarlane 1966), but less convenient for the present applications.) For a complex field $\chi^{\mu\nu}$ the number of coefficients could be greatly increased. However, if the complexity is associated with electrical charge then invariance under gauge transformations of the first kind limits us to the same real sets of 3, 4 and 6. We reject out of hand the alternative possibility of a chiral gauge group of the first kind, since it is known that electric charge is not chiral.

We adjusted the thirteen independent real coefficients in a thoroughly systematic and exhaustive way, to obtain all possible constraint chain structures leaving, in the end, a total of $2(2s+1)$ ($s=\frac{3}{2}$) independent (real or complex) pieces of Cauchy initial-value data per space point. Positivity is of course automatic in such a programme, since the little-group representation is irreducible.

In carrying out the parameter search we simplified the constraint chain analysis by using the rest frame method (as in § 2). This is always legitimate for massive fields, and it provides spin information in an immediately accessible form, so that there is no danger of inadvertently bringing in multiple spin-$\frac{1}{2}$ systems with the same data count.

The calculations were exceedingly laborious, since many of the alternative determinantal options proved to be equivalent under relative rescaling of the three irreducible constituents of $\chi^{\mu\nu}$. After discarding all such redundant possibilities we came up in the end with two distinct types of equation, each of which can exist in four distinct forms. We now give a brief description of these.

3.1. *Type 1: the decorated Rarita–Schwinger equation*

Define $(\chi^{\mu\nu})_{\rm irr}$ by equation (9) and ψ^μ by equation (10), and take as Lagrangian density the real expression

$$\mathscr{L} = \mathrm{i}am^{-1}[m\bar{\chi}^{\mu\nu} - b(\partial^\mu\bar{\psi}^\nu - \partial^\nu\bar{\psi}^\mu)]_{\rm irr}[m\chi_{\mu\nu} - b(\partial_\mu\psi_\nu - \partial_\nu\psi_\mu)]_{\rm irr}$$
$$-\tfrac{1}{2}\mathrm{i}\bar{\psi}_\mu[(\gamma\partial)\psi^\mu - 2\partial^\mu(\gamma\psi) - \gamma^\mu(\gamma\partial)(\gamma\psi) + m\psi^\mu + m\gamma^\mu(\gamma\psi)], \tag{12}$$

where a and b are real constants, and $a \neq 0$.

Since the irreducible part $(\chi^{\mu\nu})_{\rm irr}$ and the reduced part $\psi^\mu \equiv \gamma_\rho\chi^{\mu\rho}$ are kinematically independent, we may vary the former first. This splits off the Euler–Lagrange equation

$$(\chi^{\mu\nu})_{\rm irr} = m^{-1}b(\partial^\mu\psi^\nu - \partial^\nu\psi^\mu)_{\rm irr}. \tag{13}$$

We can now vary ψ^μ under this condition on $(\chi^{\mu\nu})_{\rm irr}$. The first part of the Lagrangian and all its variations are then annulled, and we obtain the Rarita–Schwinger equation (1) for ψ^μ. We recognise four cases. Two correspond to whether or not the coefficient b is zero. The other two arise through an option we have to add to $\mathscr{L}$ a constant multiple of the expression

$$\mathrm{i}[(\partial\bar{\psi})(\partial\psi) - \partial^\mu(\bar{\psi}\gamma)\,.\,\partial_\mu(\gamma\psi) + 2(\partial\bar{\psi})(\gamma\partial)(\gamma\psi)]. \tag{14}$$

The effect of exercising this option is to introduce the Glass pathology (Glass 1971, Mathews *et al* 1978), namely a tertiary stage in the spin-$\frac{1}{2}$ constraint sector.

The type of equation here described is not altogether unknown (Guralnik and Kibble 1965, Salam *et al* 1965). But since it contains a Rarita–Schwinger subsystem it is useless for our purposes.

3.2. *Type 2: a new forty-component Gel'Fand–Yaglom equation*

The second type of second-order equation which we found is by no means so easily comprehended as that given above. We may perhaps best present it in Gel'Fand–Yaglom form, i.e. as an equation of the first order. This necessitates the introduction of an auxiliary vector–spinor ξ^μ as well as the tensor–spinor $\chi^{\mu\nu}$, so that altogether we have forty (real or complex) components. The Lagrangian density is then

$$\mathscr{L} = -\tfrac{1}{2}\mathrm{i}a\bar{\chi}^{\mu\nu}[\partial_\mu - \tfrac{1}{2}m(2-a)\gamma_\mu]\xi_\nu - \tfrac{1}{4}\mathrm{i}am\bar{\xi}^\mu[\xi_\mu + \gamma_\mu(\gamma\xi)]$$
$$-\tfrac{1}{16}\mathrm{i}a^2(2-a)m\bar{\chi}^{\mu\nu}(\chi_{\mu\nu} - 2\gamma_\mu\gamma^\rho\chi_{\nu\rho} - \tfrac{1}{2}\gamma_\mu\gamma_\nu\gamma^\rho\gamma^\sigma\chi_{\rho\sigma}), \tag{15}$$

where a is a real parameter ($0 \neq a \neq 2$), and $\chi^{\mu\nu} \equiv -\chi^{\nu\mu}$ and ξ^μ are to be independently varied. The variation of ξ^μ yields the Euler–Lagrange equation

$$m[\xi^\mu + \gamma^\mu(\gamma\xi)] = \partial_\rho\chi^{\rho\mu} - \tfrac{1}{2}m(2-a)\gamma_\rho\chi^{\rho\mu}, \tag{16}$$

which fixes ξ in terms of χ and its first derivatives. The variation of $\chi^{\mu\nu}$ yields the Euler–Lagrange equation

$$ma(2-a)(\chi^{\mu\nu} - \gamma^\mu\gamma_\rho\chi^{\nu\rho} + \gamma^\nu\gamma_\rho\chi^{\mu\rho} - \tfrac{1}{4}[\gamma^\mu, \gamma^\nu]\gamma_\rho\gamma_\sigma\chi^{\rho\sigma})$$
$$= 2(\partial^\nu\xi^\mu - \partial^\mu\xi^\nu) + m(2-a)(\gamma^\mu\xi^\nu - \gamma^\nu\xi^\mu), \tag{17}$$

which fixes χ in terms of ξ and its first derivatives. We may recover our second-order equation for $\chi^{\mu\nu}$ and the second-order Lagrangian appropriate thereto by using (16) to eliminate ξ^μ from (17) and (15) respectively. But alternatively we may, if we please, use (17) to eliminate $\chi^{\mu\nu}$ from (16) and (15). When this is done, we find to our dismay that the Lagrangian cancels down to the first-order Rarita–Schwinger Lagrangian for the auxiliary field ξ^μ, and the equation of motion (16) cancels down to the Rarita–Schwinger equation for ξ^μ. By this argument we see that the type-2 second-order equation for $\chi^{\mu\nu}$ contains a standard Rarita–Schwinger constraint chain inside itself.

Thus our search has not yielded any useful new constraint structure. Nevertheless, the type-2 equation displays some interesting features, which help us to assess the situation. We distinguish altogether four cases. The most remarkable is obtained by setting $a = 1$. The second-order equation for $\chi^{\mu\nu}$ then yields all the equations of Weinberg's (1964) $(\frac{3}{2}, 0)\oplus(0, \frac{3}{2})$ neutral theory, but now in Lagrangian form. We have

$$\gamma_\rho\chi^{\mu\rho} = 0, \qquad (\gamma\partial + m)\partial_\rho\chi^{\mu\rho} = 0, \tag{18}$$

$$m^2\chi^{\mu\nu} = 2(\partial^\nu\partial_\rho\chi^{\rho\mu} - \partial^\mu\partial_\rho\chi^{\rho\nu})_{\mathrm{irr}}. \tag{19}$$

The second case ($a \neq 1$) is characterised by the presence of a non-zero reduced part $\gamma_\rho\chi^{\mu\rho}$ within $\chi^{\mu\nu}$. The reduced part does not represent any extra modes, since it is constrained to be proportional to $\partial_\rho(\chi^{\mu\rho}{}_{\mathrm{irr}})$. The remaining two cases correspond to the possibility of inducing a Glass pathology in the spin-$\frac{1}{2}$ constraint sector of the second-order tensor–spinor equation by adding to $\mathscr{L}$ a non-zero multiple of

$$\mathrm{i}\bar{\phi}(\phi + \partial_\mu\gamma_\nu\chi^{\mu\nu}), \tag{20}$$

where ϕ is a spinor to be independently varied. Equivalently, we may add a non-zero multiple of the second-order term

$$(\bar{\chi}^{\mu\nu}\overleftarrow{\partial}_\mu\gamma_\nu)(\partial_\rho\gamma_\sigma\chi^{\rho\sigma}). \tag{21}$$

Another interesting variation of the type-2 equation is to split $\chi^{\mu\nu}$ according to the scheme defined by equation (9) and to make in $\mathscr{L}$ the change of variables

$$\begin{aligned} &\chi^{\mu\nu} \to \chi'^{\mu\nu} = (\chi^{\mu\nu})_{\mathrm{irr}} - \gamma^{\mu}\xi^{\nu} + \gamma^{\nu}\xi^{\mu}, \\ &\xi^{\mu} \to \xi'^{\mu} = \gamma_{\rho}\chi^{\mu\rho}. \end{aligned} \tag{22}$$

Doing this, and taking the case $a = 1$, we are then led to the system of equations

$$\begin{aligned} &\xi'^{\mu} = 0, \qquad \gamma_{\mu}\gamma_{\nu}\chi'^{\mu\nu} = 0, \\ &\partial^{\mu}\chi'^{\nu\rho} + \partial^{\rho}\chi'^{\mu\nu} + \partial^{\nu}\chi'^{\rho\mu} = 0, \\ &(\gamma\partial + m)\chi'^{\mu\nu} = 0. \end{aligned} \tag{23}$$

Thus by introducing the auxiliary vector–spinor ξ'^{μ} we have obtained an action principle for one of the sets of non-Lagrangian tensor–spinor equations given by Fisk and Tait (1973). But the Lagrangian form is achieved only at the cost of introducing also a Rarita–Schwinger subsystem for the dynamics of $\gamma_{\rho}\chi'^{\mu\rho}$, as is clear from our derivation.

4. Dimension and the little-group classification

Some representations of the Lorentz group are demonstrably well adapted to carrying the dynamics in relativistic free-field Lagrangian systems obeying the non-negativity axioms, while others seem to be singularly useless for this purpose. This lack of adaptability seems rather strange and puzzling. Fierz and Pauli encountered and recognised the same problem in their famous paper of 1939, where the spin-$\frac{3}{2}$ Lagrangian was first written down ('we have been unable to generalise the field equations which ... correspond to $k \neq l$, or ... for which $q > 1$'). Dirac (1974) has again noted the problem ('there are some rather obscure representations of the Lorentz group which don't seem to be of much interest to physicists'). In the present paper we have tried uncommonly hard to harness the 'obscure' representations $(\frac{3}{2}, 0)$ and $(0, \frac{3}{2})$ and to give them a dynamical role, but we cannot claim any real success, since the new systems which we have found (namely (12) and (15)) are really just the Rarita–Schwinger system overlaid by extra complications. Yet our thirteen-parameter search has been rather widely formulated; in a Gel'Fand–Yaglom framework it would correspond to allowing many unusual contributions, including even some from the representations $(2, \frac{1}{2})$, $(\frac{3}{2}, 1)$, and their conjugates.

We shall now show that the mathematical potentialities of the obscure representations can be fully realised only by going up into space–times of dimensions higher than four. One of the several virtues of the method of tensor-indexed Dirac spinors becomes particularly apparent here, for it permits an obvious and immediate extension into arbitrary dimension, thus facilitating our study. Indeed, all the two-valued finite-dimensional representations of the Lorentz group in Minkowski space–times of D dimensions can be expressed in terms of tensor-indexed Dirac spinors, and their properties studied thereby (cf Murnaghan 1938).

Let us warm up by considering the Rarita–Schwinger equation (1) in D space–time dimensions. When $D = 1$ it cancels down to $0 = 0$, and is empty. When $D = 2$ it leads to $\psi^{\mu} = 0$, because its kinetic term vanishes due to the Dirac-matrix identity

$$\gamma_{\mu}\gamma_{\nu}\gamma_{\rho} + g_{\mu\nu}\gamma_{\rho} - g_{\mu\rho}\gamma_{\nu} + g_{\nu\rho}\gamma_{\mu} \equiv 0 \qquad (D \leqslant 2). \tag{24}$$

It comes to life when $D \geqslant 3$, and it then describes a particle whose little-group transformation property is that of an irreducible vector–spinor, characterised by the rest-frame structure $\psi^0 = 0$, $\gamma_i \psi^i = 0$. Clearly its curious behaviour for $D < 3$ can be set into correspondence with the fact that the said little-group structure does not exist when the space dimension $D - 1$ is smaller than 2.

Consider now the Lagrangian tensor–spinor equation ($\chi^{\mu\nu} = -\chi^{\nu\mu}$)

$$\begin{aligned}(\gamma\partial + m)\chi^{\mu\nu} &+ \gamma^\mu(\gamma\partial - m)\gamma_\rho\chi^{\nu\rho} - \gamma^\nu(\gamma\partial - m)\gamma_\rho\chi^{\mu\rho} \\ &- \tfrac{1}{4}[\gamma^\mu, \gamma^\nu](\gamma\partial + m)\gamma_\rho\gamma_\sigma\chi^{\rho\sigma} + \partial^\mu\gamma_\rho\gamma^{\nu\rho} - \partial^\nu\gamma_\rho\chi^{\mu\rho} \\ &- \gamma^\mu\partial_\rho\chi^{\rho\nu} + \gamma^\nu\partial_\rho\chi^{\rho\mu} + \tfrac{1}{2}(\gamma^\nu\partial^\mu - \gamma^\mu\partial^\nu)\gamma_\rho\gamma_\sigma\chi^{\rho\sigma} - \tfrac{1}{2}[\gamma^\mu, \gamma^\nu]\partial_\rho\gamma_\sigma\chi^{\rho\sigma} = 0.\end{aligned} \tag{25}$$

In a general dimension this epitomises the sort of thing we could have been looking for, in so far as it is indeed a reasonably efficient equation using an antisymmetric tensor–spinor. But it only comes to life when $D \geqslant 5$! Its little-group transformation property is that of the irreducible antisymmetric tensor–spinor, characterised by the rest-frame structure $\chi^{0j} = 0$, $\gamma_j\chi^{jk} = 0$. This structure cannot exist for $D < 5$, i.e. for space dimension $D - 1 \leqslant 3$, because in three and fewer dimensions the Dirac matrices obey the identity

$$\begin{aligned}\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta &+ \gamma^\nu\gamma_\beta\delta^\mu_\alpha - \gamma^\mu\gamma_\beta\delta^\nu_\alpha - \gamma^\nu\gamma_\alpha\delta^\mu_\beta \\ &+ \gamma^\mu\gamma_\alpha\delta^\nu_\beta - \tfrac{1}{4}[\gamma^\mu, \gamma^\nu][\gamma_\alpha, \gamma_\beta] \equiv 0 \qquad (D \leqslant 3),\end{aligned} \tag{26}$$

which can be used to express χ^{jk} in terms of its own contractions. The equation (25) adroitly reacts to this situation by losing its kinetic term at $D = 4$, through the identity (11)! At $D = 3$ it loses its potential term too, through the identity (26). Now the structure of the equation (25) is uniquely determined—there is nothing arbitrary about it apart from the matter of trivial relative rescalings of the three irreducible constituents of $\chi^{\mu\nu}$—and therefore these dimensional effects must be reckoned as an intrinsic and significant part of the overall mathematical situation. It thus becomes quite clear that our inability to find a sensible way to exploit $\chi^{\mu\nu}$ at the Lagrangian level is due mainly, and perhaps wholly, to the fact that we are obliged to work with $D = 4$.

Not all relativistic equations react to low dimension in quite so decisive a way, however. One might consider for example the Takahashi–Palmer equations (Takahashi and Palmer 1970, MacFarlane and Tait 1972)

$$\begin{aligned}&G^{\mu\nu\rho} - \partial^\mu H^{\nu\rho} - \partial^\rho H^{\mu\nu} - \partial^\nu H^{\rho\mu} = 0, \\ &\partial_\rho G^{\mu\nu\rho} + m^2 H^{\mu\nu} = 0, \qquad H^{\mu\nu} = -H^{\nu\mu},\end{aligned} \tag{27}$$

which have the little-group character of an antisymmetric tensor of the second rank obeying $H^{0i} = 0$ in the rest frame. At $D = 4$ these equations show a little-group redundancy with the Proca–Duffin–Kemmer equations

$$m^2 A^\mu + \partial_\nu F^{\mu\nu} = 0, \qquad \partial^\mu A^\nu - \partial^\nu A^\mu + F^{\mu\nu} = 0, \tag{28}$$

with their vector ($A^0 = 0$) little-group character. The two sets of equations are in fact mutually dual (via $\epsilon^{\mu\nu\rho\sigma}$) and therefore dynamically equivalent at $D = 4$, and neither dies at this stage. Equations (27) survive down to $D = 3$, in which dimension they become dual to the spinless Klein–Gordon–Duffin–Kemmer equation and dynamically equivalent thereto. Equations (28) are also alive in $D = 3$, but carry there a non-zero spin. They survive to $D=2$, where they in turn become dual to the spinless Klein–Gordon–Duffin–Kemmer equation and dynamically equivalent to it. It thus seems rather clear that the fate of most relativistic equations is to die at low dimension, though for some death is preceded by redundancy. The Klein–Gordon–Duffin–Kemmer and the Dirac equations are the only distinct survivors in $D=2$ and still more so in $D=1$, since in these low dimensions there is no room at all for rotational little-group concepts.

What we have just seen might tempt us to infer that little-group redundancy is always accompanied by dynamical equivalence. If this could really be shown to be so, then of course our whole search concept would be vitiated from the start. But in fact there already exists at least one counter-example—a theory containing spin $\frac{3}{2}$ yet not dynamically equivalent to any Rarita–Schwinger scheme. This counter-example is analysed in the next section.

5. The Harish-Chandra equations

Cox (1977) has recently drawn attention to the very remarkable compound-spin equations of Harish-Chandra (1947), and has studied their electromagnetic coupling. The equations can be written in any dimension higher than one. They involve a vector–spinor ξ^μ and an antisymmetric tensor–spinor $\chi^{\mu\nu}$. The Lagrangian looks quite an easy one, compared to some. It is

$$\mathscr{L}=\tfrac{1}{2}\mathrm{i}\bar{\xi}^\mu(\gamma\partial+m)\xi_\mu+\tfrac{1}{4}\mathrm{i}\bar{\chi}^{\mu\nu}(\gamma\partial-m)\chi_{\mu\nu}-\mathrm{i}\bar{\xi}^\mu\partial^\nu\chi_{\mu\nu}, \tag{29}$$

and the field equations are

$$\begin{aligned}(\gamma\partial+m)\xi^\mu&=\partial_\nu\chi^{\mu\nu},\\(\gamma\partial-m)\chi^{\mu\nu}&=\partial^\mu\xi^\nu-\partial^\nu\xi^\mu.\end{aligned} \tag{30}$$

Primary and secondary constraints arise, in consequence of which the terms on the right-hand side in the two members of (30) are both annulled. The anticommutators are positive. The equations realise their full potential for $D\geqslant 5$. They then describe four irreducible spin multiplets, degenerate in mass. The four little-group structures are characterised by

$$\begin{array}{llll}\text{(i)}\ \chi^{0j}=0, & \gamma_j\chi^{jk}=0, & \xi^\mu=0, & \\ \text{(ii)}\ \chi^{0j}=0, & \chi^{jk}=\gamma^j\zeta^k-\gamma^k\zeta^j, & \gamma_j\zeta^j=0, & \xi^\mu=0,\\ \text{(iii)}\ \chi^{0j}=0, & \chi^{jk}=[\gamma^j,\gamma^k]\eta, & \xi^\mu=0, & \\ \text{(iv)}\ \chi^{\mu\nu}=0, & \xi^j=0. & & \end{array} \tag{31}$$

In four dimensions the spin type (ii) lives as spin $\frac{3}{2}$, and types (iii) and (iv) live as spin $\frac{1}{2}$. But the spin type (i) dies, since the defining equations for it lead via identity (26) to the complete vanishing of $\chi^{\mu\nu}$. Thus it becomes relevant to ask whether the irreducible tensor spinor part of $\chi^{\mu\nu}$ actually influences the dynamics of (30) in the case $D=4$. We shall now produce a transformation to show that in fact it does not! It is present only in a 'decorative' way, as in the equations of § 3.1.

Working with $D=4$, let us introduce the irreducible part $\Omega^{\mu\nu}$ of $\chi^{\mu\nu}$ by writing

$$\chi^{\mu\nu}=\Omega^{\mu\nu}+\tfrac{1}{2}\gamma^\mu(\phi^\nu-\xi^\nu)-\tfrac{1}{2}\gamma^\nu(\phi^\mu-\xi^\mu),\qquad \gamma_\nu\Omega^{\mu\nu}=0, \tag{32}$$

where ϕ^μ is a new field introduced to carry the two reduced parts of $\chi^{\mu\nu}$. Let us also decompose ξ^μ by writing

$$\xi^\mu = \omega^\mu + \gamma^\mu \psi, \qquad \gamma_\mu \omega^\mu = 0. \tag{33}$$

Finally, let us introduce an independent auxiliary spinor field ζ, and add to the Lagrangian a term

$$\tfrac{2}{3}\mathrm{i}m^{-1}\bar{\zeta}[\tfrac{1}{2}\zeta - (\gamma\partial)(\gamma\phi) + \tfrac{1}{2}(\partial\phi)] + \tfrac{1}{3}\mathrm{i}m^{-1}[(\bar{\phi}\gamma)(\gamma\overleftarrow{\partial}) - \tfrac{1}{2}(\bar{\phi}\overleftarrow{\partial})][(\gamma\partial)(\gamma\phi) - \tfrac{1}{2}(\partial\phi)]. \tag{34}$$

The last step makes no difference at all to the dynamics, since variation of ζ produces an Euler–Lagrange equation through which the added piece and all its other variations are immediately annulled, without interposition of any constraint analysis.

After making these various changes of variable and adding the harmless extra piece we invoke the four-dimensional identity (11) to establish that the sum of the cyclic permutations of $\gamma_\mu \Omega_{\nu\rho}$ vanishes. The Lagrangian then boils down to

$$\begin{aligned}\mathscr{L} = -\tfrac{1}{4}\mathrm{i}m^{-1}(m\bar{\Omega}^{\mu\nu} - \bar{Y}^{\mu\nu})(m\Omega_{\mu\nu} - Y_{\mu\nu}) + \tfrac{1}{4}\mathrm{i}m^{-1}(m\bar{\omega}^\mu - \bar{y}^\mu)(m\omega_\mu - y_\mu)\\ -\tfrac{1}{2}\mathrm{i}\bar{\phi}^\mu(\gamma\partial + m)\phi_\mu - \tfrac{3}{4}\mathrm{i}(\bar{\phi}\gamma)(\partial\phi) - \tfrac{1}{8}\mathrm{i}(\bar{\phi}\gamma)(\gamma\partial)(\gamma\phi)\\ + \tfrac{1}{16}\mathrm{i}m(\bar{\phi}\gamma)(\gamma\phi) - \mathrm{i}\bar{\phi}^\mu\partial_\mu\psi - \mathrm{i}(\bar{\phi}\gamma)(\gamma\partial)\psi + \tfrac{3}{2}\mathrm{i}m(\bar{\phi}\gamma)\psi\\ + 2\mathrm{i}\bar{\psi}(\gamma\partial - \tfrac{1}{2}m)\psi + \tfrac{2}{3}\mathrm{i}m^{-1}\bar{\zeta}[\tfrac{1}{2}\zeta - (\gamma\partial)(\gamma\phi) + \tfrac{1}{2}(\partial\phi)],\end{aligned} \tag{35}$$

where

$$\begin{aligned}Y^{\mu\nu} &\equiv (\partial^\nu\phi^\mu - \partial^\mu\phi^\nu)_{\mathrm{irr}},\\ y^\mu &\equiv -[\partial^\mu(\gamma\phi) + (\gamma\partial)\phi^\mu + m\phi^\mu]_{\mathrm{irr}}.\end{aligned} \tag{36}$$

It follows from the derivation that the fields $\Omega^{\mu\nu}$, ω^μ, ϕ^μ, ψ and ζ are to be independently varied, subject of course to $\gamma_\mu\Omega^{\mu\nu} = 0 = \gamma_\mu\omega^\mu$. Now by inspection it is clear that the variation of $\Omega^{\mu\nu}$ and ω^μ in (35) will serve merely to fix the variables being varied, and once they are so fixed the parts of the Lagrangian in which they appear will contribute nothing to the remaining Euler–Lagrange equations. The fields $\Omega^{\mu\nu}$ and ω^μ are thus present only in a decorative and irrelevant capacity, and contribute nothing to the dynamics. Once again, we are forced to the conclusion that the dynamics of spin $\frac{3}{2}$ just *has* to be carried by a vector–spinor! That the ϕ^μ, ψ, ζ system contains a double complement of Dirac spinors in addition to the usual vector–spinor seems fully consonant with the fact that it describes two multiplets of spin $\frac{1}{2}$ in addition to one of spin $\frac{3}{2}$.

This example thus serves once again to suggest that it is mathematically impossible to make Lagrangian theories obeying the positivity axioms and having the representations $(\frac{3}{2}, 0)$ and $(0, \frac{3}{2})$ at the heart of their dynamics.

But there is one more surprise in store. Although the ϕ^μ, ψ, ζ system describes three multiplets of spins $\frac{3}{2}$, $\frac{1}{2}$, $\frac{1}{2}$ it is not possible to split off from it a Rarita–Schwinger spin-$\frac{3}{2}$ subsystem. For the only vector–spinor available, namely ϕ^μ, carries one of the spin-$\frac{1}{2}$ modes in its time part and the other in its space part, in such a way that no local linear combination of the form

$$\Psi^\mu \equiv \phi^\mu + \gamma^\mu[a\psi + b\zeta + c(\gamma\phi)] \tag{37}$$

can ever be rendered free of both. The primary and secondary constraints in the zero-momentum frame are respectively

$$\gamma_0\phi^0 + \tfrac{2}{3}m^{-1}\zeta = 0, \qquad \gamma_j\phi^j + 12\psi - 2m^{-1}\zeta = 0, \tag{38}$$

which make it impossible even to achieve the Rarita–Schwinger conditions $\Psi^0 = 0 = \gamma_j \Psi^j$ for any choice of a, b, c, let alone to dynamically decouple Ψ^μ. Undoubtedly this is connected with Harish-Chandra's assertion that his system of equations is irreducible, though the connection is probably not straightforward since the first two terms in the transformed system (35) are not in Gel'Fand–Yaglom form.

6. Discussion and conclusions

In § 1 we have given a very simple argument, according to which the failure of causality in interacting high-spin theory stems directly from the presence of secondary constraints in the Cauchy problem. In § 2 we have re-examined the theorem of Gel'Fand and Yaglom from the viewpoint that this suggests, and have proved that secondary constraints are an inevitable concomitant of non-negativity for any spin value higher than 1. We have proved this without making any assumptions whatever as to the nature of the mass spectrum, other than that there should be no massless modes or gauge freedoms. We have also used non-negativity to prove that all masses will be real, and that a complete dynamical decomposition into simple (BE or FD) oscillator modes will always be possible. This disposes of all theories governed by multiple Klein–Gordon or Dirac factors, etc. We have also mentioned the insuperable problem that confronts massive high-spin gauge theories, namely, that electromagnetic interaction breaks the gauge invariance.

Recognising these very stringent limitations, we have conducted an extensive thirteen-parameter search (§ 3), to see whether the Rarita–Schwinger constraint scheme is the only one which can accommodate spin $\frac{3}{2}$ and the non-negativity axiom. One of our ideas when we started was that there might well be some special sort of equation peculiar to dimension 4. This expectation seemed not unreasonable, in view of the special spinor and tensor identities, such as (11), which hold in that dimension. Our method of search would have found such peculiar equations, had they existed. But in fact the new systems that we did find, namely (12) and (15), can be extended to higher dimension almost as we have written them. All that is necessary is to adjust the values of some of the constant factors, to allow for the dimension dependence of $\gamma^\mu\gamma_\mu$. Thus the only thing that one can do with special identities such as (11) is to use them to rewrite the equations in a form which does not extend at sight to higher dimension—hardy a useful exercise! Another of our hopeful ideas when we started was that a vector–spinor Ψ^μ defined as the inner derivative $\partial_\rho \chi^{\mu\rho}$ of a γ-irreducible tensor–spinor would automatically obey both of the Rarita–Schwinger covariant conditions $\partial_\mu \Psi^\mu = 0 = \gamma_\mu \Psi^\mu$. But although our new equations (18) and (19) do embody precisely this seemingly promising feature, they still depend upon an internal Rarita–Schwinger constraint chain, as we have proved. Thus, for all its flexibility, our approach has fulfilled neither of the expectations with which we started, and has yielded no improvement over the Rarita–Schwinger equation.

In § 4 we have discussed the dimensional aspects of the problem, and have shown that our total failure to find significant new forms is due to special dimensional identities, particularly the identity (11). In this way we have resolved a longstanding puzzle, and clarified our general understanding of the negative role of some types of tensors and tensor–spinors in the theory of linear relativistic systems.

In § 5 we have analysed Harish-Chandra's compound-spin theory (spins $\frac{3}{2}$, $\frac{1}{2}$, $\frac{1}{2}$), and have drawn two significant lessons from it. Firstly, we have seen that even when the spin is compound the irreducible tensor–spinor still plays no part in the four-dimensional dynamics, and that the latter can be expressed entirely in terms of vector–spinors and ordinary spinors. Secondly, we have seen that the compound-spin constraint structure is inequivalent to that found in Rarita–Schwinger theory, in that a Rarita–Schwinger subsystem cannot be separated off by making local linear field transformations. So we see that it would be wrong to conclude that spin $\frac{3}{2}$ is necessarily accompanied by a Rarita–Schwinger structure. But we may reasonably surmise that its dynamics at least is always centred upon vector–spinors, and not upon tensor–spinors, for this is true of all the theories we have now seen. The compound-spin theory does not of course match anything found presently in the real world, and moreover Cox (1977) has shown that causality problems arise in it too. Nevertheless, the strange structure exhibited by this theory might prompt a further search for pure spin-$\frac{3}{2}$ field equations within the context of a vector–spinor supplemented by n ordinary spinors. However, it seems altogether unlikely to us that anything of physical utility could possibly arise this way, since one would need still more constraints to get rid of the superfluous spinor degrees of freedom. Hurley and Sudarshan (1975) have indeed already looked at the case $n = 1$, and have shown that the only possibilities there are the Rarita–Schwinger and the Glass (cf § 3) equation. The case $n = 2$ encompasses the Harish-Chandra equation—we have not examined it to see precisely what unusual pure-spin equations it might admit, though presumably all will be in some sense generalisations of the Glass equation, and therefore useless for physical application.

Subject to the physically necessary non-negativity axiom our discussion allows us to conclude that the irreducible tensor-spinor representations $(\frac{3}{2}, 0)$ and $(0, \frac{3}{2})$ are useless for field dynamics in four dimensions, and that all pure spin-$\frac{3}{2}$ systems display Rarita–Schwinger or Rarita–Schwinger–Glass structure or a higher-order generalisation of the latter. The same sort of thing presumably holds for spin 2 and for all other high spins, and so it follows that the only massive fields to which elementary dynamical status can be attributed are those with spins 0, $\frac{1}{2}$ and 1. It seems remarkable that a mathematical argument so devious, indirect and difficult as that we have indicated should point in the same direction as quark phenomenology.

For massless particles the same spin limitations hold, but with some special exception for gravity and, perhaps, for supergravity also. Both gravity and supergravity (Deser and Zumino 1976, Fayet and Ferrara 1977) escape the usual causality difficulties by working with conserved sources. The fundamental problem with quantum gravity, as we see it, is epistemological, in that it seems to abolish the classical background that one needs before one can set up rules for physical interpretation. Supergravity seems to bring a further problem, in that it does not possess a satisfactory stress tensor. In our 1977 paper on massless spin-$\frac{3}{2}$ theory we found a problem of gauge variance, which afflicts all local energy concepts in massless gauge theories having spin higher than one. These difficulties, which we showed to be insuperable, undoubtedly afflict the stress tensor of supergravity, at least in the flat space–time limit, and cannot be resolved on the basis of conventional wisdom†.

So we reiterate our conclusion. Out of the great plethora of mathematically possible high-spin, multi-spin, multi-mass, etc Lagrangian systems, the only ones that

† *Note added in proof.* The problem of the gauge invariance of the stress tensor has also been considered by D Z Freedman and A Das. We thank Professor Freedman for his comments and Professor Das for sending us a copy of his work (Das 1978).

are adaptable to physics are the well known irreducible equations for spins 0, $\frac{1}{2}$ and 1 and, at a somewhat different level, the Einstein–Hilbert equations for the gravitational field.

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Note added in proof. The problem of the gauge invariance of the stress tensor has also been considered by D Z Freedman and A Das. We thank Professor Freedman for his comments and Professor Das for sending us a copy of his work (Das 1978).

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